ABSTRACT

Finite forms of the two common eigenvalue problems associated with the uncertainty principle are introduced. The results are extended to two dimensions and it is shown that feature extraction filters produced using these methods are effective in the processing of natural images.

INTRODUCTION

Thanks to the work of Gabor [1] and Slepian and his co-authors [2] - [7], the uncertainty principle has become established as one of the central results in signal processing, with applications from sampling theory to spectrum estimation, filter design and image coding [8] - [11]. It has recently received attention in the literature on vision and image processing [12] - [18].

The purpose of the work reported here is two-fold. First, it extends the earlier results to finite transforms. Results are derived for the finite forms of both the harmonic oscillator and prolate spheroidal equations. Two-dimensional forms of the problems are considered. These results are the natural finite analogues of those presented by both Gabor and Slepian. By treating both problems together as eigenvalue problems, their common features may be seen more clearly. Furthermore, the use of finite transforms has conceptual and computational advantages and is, in any case, natural in image processing.

Secondly, the results of these investigations have been applied to one of the classic problems in image processing - the design of feature-extraction filters. Filters constructed using Gabor-type "signals of Gaussian envelope" and using a polar-separable form of the finite prolate-spheroidal wave equation have been used in an algorithm for image feature extraction first described by Knutsson and Granlund [10], [15]. This has proved to be of great utility in problems such as image enhancement [17] and data compression [18] and has a clearly established relation to known visual system properties [12], [13]. The results obtained with filters designed using the eigenvalue approach show that they provide a flexible alternative to those obtained with the heuristically derived filters described by Knutsson [10]. A fuller account of these ideas is contained in [19].

MATHEMATICAL PRELIMINARIES

The finite form of the uncertainty principle introduced by Gabor [1] is based on the (N x N) DFT matrix P

\[
P = \frac{1}{N} \exp(-j \pi k_1) \quad f_{kl} = N^{-1} \exp(-j \pi k_1) \delta_{kl}, \quad 1 \leq k \leq N
\]

and let \( \Sigma^2 \) be the discrete variance operator

\[
\Sigma^2 = (\delta^2)_{ij}
\]

\[
= N^{-1} \delta_{ij} \quad i \leq N
\]

Then the discrete harmonic oscillator problem can be written as

\[
H\phi_i = (\Sigma^2 + P^* \Sigma^2 P) \phi_i = \lambda_i \phi_i \quad 0 \leq i < N
\]

Some straightforward but tedious calculation allows the explicit formulation of the operator H

\[
H = (h_{ij})
\]

\[
h_{ij} = \begin{cases} (-1)^{i-j} & \text{if } i \neq j \\ \frac{2 \sin^2 \pi(i-j)}{N} & \text{if } i = j \end{cases}
\]

The second form of the uncertainty principle to be considered is that due to Slepian et al [2] - [7]. Let \( T_n \) be the truncation operator

\[
T_n = \begin{cases} 1 & 0 \leq i < n \\ 0 & \text{else} \end{cases}
\]

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and let $T_n$ be the symmetric truncation operator

\[ T_n = T \cdot n + 1 \cdot S^{-1}(n-1)T \cdot n \cdot (n-1) \]

(6)

where $S(i)$ is the $i$th shift operator. Now define $W_m,n$ as

\[ W_{m,n} = T \cdot F \cdot T \cdot F \cdot W \]

(7)

Then the finite form of the discrete prolate-spheroidal wave equation is simply

\[ m,n \]

(8)

where the eigenvalue $\lambda_i$ represents the fraction of energy remaining after the spatially limited vector $\phi_i$ is bandlimited, since

\[ \langle \phi_i, W_{m,n}, \phi_i \rangle = \langle \phi_i, F \cdot T \cdot F \cdot \phi_i \rangle = \lambda_i \langle \phi_i, \phi_i \rangle \]

(9)

Explicit calculation of $W_{m,n}$ yields

\[ W_{m,n} = \lambda_i \cdot \delta_{ij} + S \cdot \delta_{ji} \cdot \sum_{i < j < n} \]

(10)

a form which bears an obvious similarity to that discussed by Slepian in [7].

The final question to consider is the transition from one to two dimensions. The simplest method of achieving this is of course to use cartesian separable operators, that is operators $M$ which can be written as a Kronecker product:

\[ M = M_h \otimes M_v \]

(11)

where $M_h$ and $M_v$ are the matrices of the corresponding 1-d operators. This leads directly to a solution of the eigenvalue problem

\[ M = \lambda \cdot \phi \]

(12)

as

\[ M_h \cdot \phi_h = \lambda_h \cdot \phi_h \]

(13)

where $\lambda_h, \lambda_v$ are the eigenvalues of the 1-d problems associated with the eigenvectors $\phi_h$ and $\phi_v$ respectively and

\[ \phi_i = \begin{bmatrix} h \cdot \phi_h \\ v \cdot \phi_v \end{bmatrix} \]

(14)

is the $(N^2 \times 1)$ product eigenvector.

In areas such as image processing, however, there is frequently a desire for operators which are either circularly symmetric or polar separable. It is well known that the cartesian separable Gaussian signal has the property of circular symmetry in two dimensions. While its discrete counterparts, as defined in Eqs. (1) - (4) are not perfect in this respect, they are nonetheless satisfactory in many cases, as will be shown in the experimental section below. The prolate spheroidal functions were generalized to spaces of 2 or more dimensions and arbitrary region shapes in [4]. In the discrete case, it is also possible to define arbitrary region shapes in 2-d in either domain, or both. One form of some interest is to maintain cartesian separability in one domain, but to define the truncation region in the other domain using polar separable functions. If, for example, a square 2-d region $X$ is defined as

\[ X = \{(i,j) : 0 \leq i < N, 0 \leq j < N\} \]

(15)

it is possible to define a segment $R$ within $X$ as

\[ R = \{(i,j) : r \cdot \min < r(i,j) < r \cdot \max, 0 \leq \theta(i,j) < \theta \cdot \max \} \]

(16)

where

\[ r(i,j) = \frac{(i-N)^2 + (N-j)^2}{2} \]

(17)

\[ \theta(i,j) = \tan^{-1} \frac{(N-j)}{(i-N/2)} \]

(18)

A two-dimensional truncation operator $T_R$ can then be written as

\[ T_R = (T_R \cdot i_j) = \begin{bmatrix} h \cdot \delta_{ij} \\ v \cdot \delta_{ij} \end{bmatrix} \]

(19)

A symmetric operator $T_{R+}$ is also readily defined by extending $R$ to

\[ R' = \{(i,j) : r \cdot \min < r(i,j) < r \cdot \max, 0 \leq \theta(i,j) < \theta \cdot \max \} \]

(20)

This leads to an eigenvalue problem of the form

\[ T \cdot F \cdot T \cdot F \cdot W \cdot T \cdot F = \lambda \cdot \phi \]

(21)

in which all of the operations save the last are cartesian separable, with $m_h$ and $m_v$ representing the horizontal and vertical bandwidths of the frequency domain truncation. Quadrature filter pairs produced using the frequency domain counterpart of Eq. (20) have been used in the image feature extraction experiments described below.

## IMAGE FEATURE EXTRACTION

A useful test of the above methods is to apply them to the design of filters for the extraction of image features such as lines and edges. This work represents both a test of the flexibility of the eigenvalue-based method and closes the gap between the heuristically derived filters discussed in [10] and the more abstract methods of Gabor and Slepian. It is, moreover, of considerable interest in the context of models of vision, since a number of papers have recently appeared which point strongly to a close link.
between such feature representations and visual system operation \[12\], \[13\].

The aim of this work is to produce a local description of image features such as lines and edges in terms of their magnitude and orientation, which together constitute a vector \( \mathbf{e}(x,y) \) at point \((x,y)\)

\[
\mathbf{e}(x,y) = (e_1(x,y), e_2(x,y))
\]

where \( \phi(x,y) \) is simply

\[
\phi(x,y) = \text{Arg}[\mathbf{e}(x,y)]
\]

The method used is based on convolution of the image \( f(x,y) \) with 4 quadrature pairs of filters, with directions separated by 45°, giving a spatially localized estimate of the energy in a given segment of the frequency domain. Thus if \( h_i(x,y) \) is the complex (i.e. quadrature pair) filter in the \( i \)th direction, the estimated energy \( b_i(x,y) \) at point \((x,y)\) is

\[
b_i(x,y) = \left| \sum_{i=1}^{l} h_i(x_y, y-y_i) \right|^2
\]

and then

\[
e_1(x,y) = b_1(x,y) - b_{i+2}(x,y)
\]

For appropriate choices of filter function, the resulting orientation estimate is unbiased. The method is fully described in \[10\].

In the experiment, the feature-extraction filters were based directly on 'uncertainty' problems. One was a signal of 'Gaussian' envelope and was cartesian separable, obtained by solution of Eq. (3) for the case \( N=32, \sigma=1 \). It was truncated to \((15 \times 15)\) elements spatially and modulated with 1-d complex exponentials of frequency \( f = \frac{1}{4 f_{\text{max}}} \) in each of the 4 directions. The other was obtained by solution of generalized prolate spheroidal equations of the form of Eq. (20), with parameters chosen to give the large angular bandwidth required for successful use of the algorithm described in Eqs. (21)-(24). This filter required a spatial window of \((11 \times 11)\) pixels.

Each set was used on the test image of Fig. 1 which is a circularly symmetric logarithmic FM pattern. The results of the tests are shown in Figs. 2-3. The results illustrate that the broadband filters of Fig. 2 perform better than the 'Gaussian' filters. Note also that the complex Gaussian filter set shows a marked phase-dependence of the output. This is a consequence of its not being an analytic signal - its components are not in perfect quadrature.

CONCLUSIONS

Discrete forms of the eigenvalue problems which form the basis of uncertainty principle have been introduced for finite transforms. The 2-d forms of the problem have been discussed. Experimental results using 2-d eigenvectors have shown that they are a useful and flexible tool for image processing. With suitable adaptation they can find application in areas as diverse as spectrum estimation \[11\] and image data compression \[18\], \[10\].

The work reported here represents part of a study of the role of uncertainty in image processing. Further application of this central principle can be found in references \[13\], \[18\] and \[19\].

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REFERENCES

9. Shanmugam, K., Didley, F. M., Green, J. A.
   An Optimal Frequency Domain Filter
   for Edge Detection in Digital Pictures
   IEEE Trans. on Pattern Anal. and

10. Knutsson, H.
    Filtering and Reconstruction in
    Image Processing
    Linkoping University, Ph.D. Thesis,
    1982

11. Thomson, D. J.
    Spectrum, Estimation and Harmonic
    Analysis
    Proc. IEEE, vol. 70, pp. 1055-1096,
    1982

12. Marcelja, S.
    Mathematical Description of the
    Responses of Simple Cortical Cells
    Jnl. Opt. Soc. America, 70, pp. 1297-
    1300, 1980

13. Wilson, R.
    The Uncertainty Principle in Vision
    Linkoping Univ. Int. Rept no
    LiTH-isy-I-0381, 1983

14. Marr, D., Hildreth, E.
    Theory of Edge Detection
    MIT, AI Memo no 518, 1979

15. Granlund, S.
    In Search of a General Picture
    Processing Operator
    Comput. Graph. and Im. Proc., 8,
    pp. 155-173, 1978

16. Wilson, R., Granlund, G. H.
    The Uncertainty Principle in Image
    Processing
    Proc. 3rd Scand. Conf. on Image Anal.,
    Copenhagen, 1983

17. Knutsson, H., Wilson, R., Granlund,
    G. H.
    Anisotropic, Non-stationary image
    estimation and its applications; I;
    image restoration
    IEEE Trans. on Commun., COM-31,
    pp. 388-406, Mar. 1983

18. Knutsson, H., Wilson, R., Granlund,
    G. H.
    Anisotropic, Non-stationary image
    estimation and its applications; II;
    predictive image coding
    IEEE Trans. on Commun., COM-31,
    pp. 388-406, 1983

19. Wilson, R.
    Uncertainty, Eigenvalue Problems and
    Filter Design
    Linkoping Univ. Int. Rept no
    LiTH-isy-I-058C, 1983

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